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Variations on the Dembowski–Wagner theorem

William M. Kantor

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

In memory of Jaap Seidel

Abstract

Let **D** be a symmetric design admitting a null polarity such that either all singular lines, or all nonsingular lines, have size $(v-\lambda)/(k-\lambda)$; assume that this number is greater than λ in the case of singular lines. Then **D** is either a projective space or an orthogonal symmetric design.

The Dembowski–Wagner Theorem [3] characterizes projective spaces as the only symmetric designs such that all lines have size $(v-\lambda)/(k-\lambda)$. This note contains two analogous characterization results concerning a symmetric design admitting a null polarity. Whereas the proof in [3] depends on classical axioms for projective spaces [8], the results below follow very easily from far more difficult theorems due to Buekenhout and Shult [2] and Hall [5] concerning polar spaces.

Recall that, for any symmetric design **D**, if x and y are distinct points then their *line* xy is defined to be the intersection of all blocks containing x and y. Two points are on exactly one line, and a line of size $(v-\lambda)/(k-\lambda)$ has nonempty intersection with every block [3]. Assume that **D** is equipped with a null polarity $x \to x^{\perp}$ (thus, $x \in x^{\perp}$ for all points x). We call a line *singular* if it contains distinct points x, y such that $y \in x^{\perp}$, and *nonsingular* otherwise. If $x \neq z \in xy \subseteq x^{\perp}$ then $x, z \subseteq z^{\perp}$ and hence $y \in xy = xz \subseteq z^{\perp}$: a line cannot be both singular and nonsingular.

In addition to the projective geometries $\mathrm{PG}(d, q)$ we will need to consider Higman's "orthogonal" symmetric designs [6], having the same parameters; its points are the singular points x of a d-dimensional orthogonal $\mathrm{GF}(q)$-space with q and d odd, and its blocks correspond to the hyperplanes $x^{\perp}$.

Theorem 1. *Let* **D** *be a symmetric design admitting a null polarity such that all singular lines have size* $(v-\lambda)/(k-\lambda) > \lambda$. *Then* **D** *is either a projective space or an orthogonal design.*

E-mail address: kantor@math.uoregon.edu (W.M. Kantor).

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Proof. Consider the geometry $\mathcal{G}$ consisting of the points of **D** and the singular lines. By hypothesis, every block $z^\perp$ has nonempty intersection with every singular line L; and if $z^\perp$ contains two points of L then it contains L. This is just the Buekenhout–Shult property for $\mathcal{G}$. Clearly no point is perpendicular to all others; and the hypothesis $(v-\lambda)/(k-\lambda) > \lambda$ implies that there are three noncollinear pairwise perpendicular points. It follows from [2] that $\mathcal{G}$ arises from a polar space.

The polar space determines the blocks of **D** since $x^\perp$ consists of the points collinear with x (with respect to $\mathcal{G}$); and hence $\mathcal{G}$ produces a symmetric design if and only if $|x^\perp \cap y^\perp|$ is the same whenever x and y are distinct. It is easy to check all polar spaces in order to see that only symplectic and odd-dimensional orthogonal ones produce symmetric designs. The designs in the symplectic case are projective spaces. □

Theorem 2. *Let* **D** *be a symmetric design admitting a null polarity such that all nonsingular lines have size* $(v - \lambda)/(k - \lambda)$. *Then* **D** *is a projective space.*

Proof. This time consider the geometry $\mathcal{G}'$ consisting of the points of **D** and the nonsingular lines. Every block $z^\perp$ has nonempty intersection with each nonsingular line L. This time, if $z^\perp \supseteq L$ then no line zu, $u \in L$, is in $\mathcal{G}'$, while if $z^\perp \cap L$ is a point w then zu is a line of $\mathcal{G}'$ for each $u \in L - \{w\}$. Thus, $\mathcal{G}'$ is a "copolar space" and is "indecomposable" and "reduced" in the sense of [5]. All possibilities for $\mathcal{G}'$ were determined in [5, Theorem 2].

Each possible $\mathcal{G}'$ determines the blocks of **D** since $x^\perp$ consists of x and the points not collinear with x (with respect to $\mathcal{G}'$); and, as above, $\mathcal{G}'$ produces a symmetric design if and only if $|x^\perp \cap y^\perp|$ is the same same whenever x and y are distinct. It is straightforward to check the geometries listed in [5, Theorem 2] in order to see that the only ones arising from symmetric designs are the geometries of points and nonsingular lines of symplectic geometries. Once again the corresponding designs are projective spaces. □

Remark 1. It is natural to wonder about theorems of the following sort: *if a symmetric design has sufficiently many lines of size* $(v - \lambda)/(k - \lambda)$ *then it must be a projective space.* How many is "sufficiently many" ? One might conjecture that it is enough to have more than half the number $L(d, q) = (q^{d+1} - 1)(q^d - 1)/(q^2 - 1)(q - 1)$ of such lines that $\mathrm{PG}(d, q)$ has. However, elementary examples obtained using the method in [7] show that there are symmetric designs with the same parameters as $\mathrm{PG}(d, q)$, $d \geq 3$, other than $\mathrm{PG}(d, q)$, having at least $(q^{d-1} - q^{d-2})q^{d-1} = L(d, q)(1 - o(1/q))$ lines of size $q + 1$.

Remark 2. These theorems were motivated by an application to a family of symmetric designs related to the groups $G_2(q)$ [4].

In memory. The involvement of Jaap Seidel in [2] was described at length in [1], including the following: "Seidel laid much pressure on me with his unique insight and skill to coach. He insisted that I got in touch with Shult and I did so. Eventually the story led to the Buekenhout–Shult 1974 theory. The team was built by Seidel". These skills are familiar to all who knew Jaap.

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